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On the asymptotic diffraction of scalar waves by a wedge†

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Abstract. The diffraction of plane, cylindrical and spherical waves by a wedge is considered. Particular emphasis is placed on finding accurate and simple asymptotic field expansions which are valid in the transition regions as well as in the far field. The accuracy of previous diffraction coefficients for cylindrical and spherical waves is discussed and a more accurate derivation is presented. A new simple rational approximation of the resulting Fresnel integral is given and its special character is demonstrated.

1. Introduction

The problem of the diffraction of waves by a wedge has been the subject of extensive theoretical studies, using several techniques. The usual types of incident scalar waves considered in these studies are plane, cylindrical and spherical waves. The electric or magnetic dipole represents the most elementary source of electromagnetic waves whose diffraction by a wedge has been treated by many contributors. Good reviews of the work on the diffraction of these waves by a half-plane or a wedge are given in Rubinowics (1966), Bowman *et al* (1969) and Mohsen (1971).

The major contribution to the theory of the asymptotic treatment of the solution, at distances far from the diffracting edge, is attributed to Pauli (1938). More recent contributions include the works by Hutchins and Kouyoumjian (1969) and Mohsen (1971). These asymptotic solutions have direct application in the theory of high-frequency diffraction by bodies of complex shapes with edges using the geometrical theory of diffraction GTD (Keller 1962). In particular, they increase the accuracy of the GTD solutions, extend the range of applicability of the theory and correct the contributions along the shadow and reflection boundaries (Kouyoumjian and Pathak 1974). In view of the availability of several forms of the exact solution (Bowman *et al* 1969), it is to be expected that the accuracy of the asymptotic development depends on the form used. A particularly convenient integral representation is used in this work and its advantage is explained.

Usually, the accuracy of the deduced asymptotic expansions for cylindrical and spherical wave excitations is not as good as the corresponding plane-wave approximations. A method is presented here which yields more accurate asymptotic expressions for these excitations and at the same time preserves the familiar expression of these expansions in terms of Fresnel's integral. Also, a particularly useful rational approximation of this integral is presented and its advantage is discussed.

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2. Integral representation of the solution

Consider a perfectly conducting wedge of exterior angle α whose edge coincides with the z axis in a cylindrical coordinate (r, ϕ, z) system. Let U^0 denote the incident field which is given by

$$U_{\mathrm{p}}^0 = \exp[-jkr\cos(\phi-\phi_0)] \qquad \text{for a plane wave} \tag{1a}$$

$$U_{\mathrm{c}}^0 = H_0^{(2)}(kr_1) \qquad \text{for a cylindrical wave} \tag{1b}$$

$$U_{\mathrm{s}}^0 = \exp(-jky)/y \qquad y^2 = r_1^2 + (z-z_0)^2 \qquad \text{for a spherical wave} \tag{1c}$$

where (r_0, ϕ_0, z_0) are the source coordinates, $r_1^2 = r^2 + r_0^2 - 2rr_0\cos(\phi-\phi_0)$ and a suppressed $\exp(j\omega t)$ time dependence is assumed. The total scalar field U due to the incident wave should satisfy the Helmholtz equation exterior to the wedge and the radiation condition in the far field beside having the correct edge singularity. A particularly convenient representation of U is given in the form (Oberhettinger 1958)

$$U = S(|\phi-\phi_0|) \mp S(|\phi+\phi_0|) \tag{2}$$

where the negative and positive signs correspond to the soft and hard boundary conditions, respectively. The function $S(\theta)$ may be written as

$$S(\theta) = S_1(\theta) + S_2(\theta) \tag{3}$$

where S_1 represents the geometrical optics contribution and S_2 is the diffraction term which may be written in the form (Oberhettinger 1958)

$$S_2(\theta) = -(2\alpha)^{-1}\int_0^\infty E(x)[V(x, \pi-\theta) + V(x, \pi+\theta)]\,\mathrm{d}x \tag{4}$$

where the excitation function $E(x)$ depends on the type of the incident field and is given by

$$E_{\mathrm{p}}(x) = \exp(-jkr\cosh x) \tag{5a}$$

$$E_{\mathrm{c}}(x) = H_0^{(2)}[k\bar{r}_1] \tag{5b}$$

or

$$E_{\mathrm{s}}(x) = \exp(-jk\bar{y})/\bar{y} \qquad \bar{y}^2 = \bar{r}_1^2 + (z-z_0)^2 \tag{5c}$$

corresponding to the three cases of excitations given by equation (1). In equation (5), $\bar{r}_1^2 = r^2 + r_0^2 + 2rr_0\cosh x$. The pattern function $V(x, \delta)$ in equation (4) is given by

$$V(x, \delta) = \sin(\pi\delta/\alpha)/[\cosh(\pi x/\alpha) - \cos(\pi\delta/\alpha)]. \tag{6}$$

This expression may also be written in the form (Whipple 1917)

$$V(x, \delta) = \sum_{m=1}^{\infty} \exp(-m\pi x/\alpha)\sin(m\pi\delta/\alpha). \tag{7}$$

3. Accuracy of the asymptotic expansion

For large kr, the stationary-phase method may be employed in order to find the asymptotic expansion of the diffraction integral. The stationary-phase point is at $x = 0$. It is important to note that $V(x, \delta)$ decays as x increases and has an exponential decay rate in the form $\exp(-\pi x/\alpha)$. Consequently, the effective range of x in equation (4) is at small values of x, especially for larger (π/α). Thus, the major contribution is from the neighbourhood of $x=0$ even in the near field. This is the reason for the high accuracy of the stationary phase result derived from the chosen solution form (Mohsen 1971) compared with the exact series solutions for plane-wave and cylindrical-wave excitations.

Usually, the accuracy of the asymptotic expressions derived for cylindrical- and spherical-wave incidence is not as good as the corresponding plane-wave approximations. The reason is due to the asymptotic expansion which is usually performed for $E(x)$ for the first two wave types prior to the application of the stationary-phase method. A more accurate approximation may be obtained following a particular type of expansion (Mohsen and Shafai 1981) which proved to be effective in similar situations. In particular, the function $E(x)$ is expanded for small values of x in the form $\exp(-\psi x^2)$ where ψ is to be determined in order to increase the accuracy of the expansion.

Thus, for cylindrical wave incidence, let

$$g(x)=H_0^{(2)}[k(r^2+r_0^2+2rr_0\cosh x)^{1/2}]\exp(\psi x^2). \tag{8}$$

Expanding $g(x)$ for small x, one obtains

$$g(x)=H_0^{(2)}[k(r+r_0)]+\frac{1}{2}\left(H_0^{(2)}[k(r+r_0)]-\frac{krr_0\psi}{2(r+r_0)}H_1^{(2)}[k(r+r_0)]\right)x^2+\{\ldots\}x^4\ldots. \tag{9}$$

Then, the coefficient of x^2 vanishes upon taking

$$\psi=\frac{krr_0}{[2(r+r_0)]}\frac{H_1^{(2)}[k(r+r_0)]}{H_0^{(2)}[k(r+r_0)]}. \tag{10}$$

A further approximation may be performed upon invoking the assumption that $k(r+r_0)\gg 1$. In this case, equation (10) yields

$$\psi\simeq jkrr_0/[2(r+r_0)]. \tag{11}$$

The diffraction term $S_2(\theta)$ may be written in the form

$$S_2(\theta)=I(\pi-\theta)+I(\pi+\theta) \tag{12}$$

where

$$I(\delta)=-(2\alpha)^{-1}\int_0^\infty E_c(x)V(x,\delta)\,\mathrm{d}x. \tag{13}$$

Upon neglecting terms higher than x^2 in equation (9), we get from equations (8) and (9)

$$E_c(x)\simeq H_0^{(2)}[k(r+r_0)]\exp(-\psi x^2) \tag{14}$$

where ψ is given by equation (10) or equation (11). The introduction of this approximation in equation (13) and evaluation of the resulting integral yield

$$I(\delta)\simeq-\cos t\ \mathrm{sgn}\ t\ \exp(j\xi^2+j\pi/4)H_0^{(2)}[k(r+r_0)]F(\xi)/\sqrt{\pi} \tag{15}$$

where

$$t=\pi\delta/(2\alpha)\qquad \mathrm{sgn}\ t=\sin t/|\sin t|\qquad \xi=\alpha|\sin t|[2krr_0/(r+r_0)]^{1/2}/\pi$$

ψ is given by equation (11), and the Fresnel integral $F(\xi)$ is defined by

$$F(\xi)=\int_{\xi}^{\infty} \exp(-jt^2)\,\mathrm{d}t. \qquad (16)$$

A similar analysis may be performed for spherical wave incidence. In this case, $E_s(x)$ is approximated by

$$E_s(x)\simeq[\exp(-jka)/a]\exp[-jkrr_0/(2a)] \qquad (17)$$

and the corresponding diffraction integral is approximated by

$$I(\delta)\simeq -\cos t\ \mathrm{sgn}\ t\ \exp(j\beta^2+j\pi/4)[\exp(-jka)/a]F(\beta)/\sqrt{\pi} \qquad (18)$$

where $a^2=(r+r_0)^2+(z-z_0)^2$ and $\beta=\alpha|\sin t|(2krr_0/a)^{1/2}/\pi$.

Since $F(0)=\sqrt{\pi}\exp(-j\pi/4)/2$, the above expansions are finite along the shadow and reflection boundaries. Besides, the discontinuity in the geometrical optics field at these boundaries is compensated by that in $I(\delta)$ due to the discontinuity in sgn(t) at $t=\delta=0$.

To test the accuracy of our approximate formulae, we compare the results with the series solution available for an H-polarised line source excitation (Bowman *et al* 1969). For a wedge with $\alpha=200°$, $\phi_0=20°$, $kr_0=1$ and $\phi=10°$, the error in both the field amplitude and phase is less than 5% for $kr\geqslant 9$.

4. Rational approximation of the Fresnel integral

The frequent appearance of the Fresnel integrals in asymptotic diffraction theory, and their particular use at shadow and reflection boundaries, justify seeking a simple particular approximation for these integrals. Such an approximation is required to give reasonably accurate values at very small arguments, corresponding to the shadow and reflection boundaries, as well as for large arguments which are encountered in far-field analysis.

Using the leading terms in the small and the large argument expansions of the Fresnel integral, and with the above objective in mind, a simple rational approximation may be written in the form

$$F(\xi)\simeq[\sqrt{\pi}\exp(-j\pi/4)/2-\xi\exp(-j\xi^2)]/(1-2j\xi^2). \qquad (19)$$

This approximation yields the exact values at $\xi=0$ and gives the correct leading asymptotic term as $\xi\to\infty$. When this approximate expression is compared with the tabulated values, the error is found to be less than 11% for $\xi<0.2$ and $\xi\geqslant 10$.

5. Discussion and conclusions

The asymptotic diffraction of plane, cylindrical and spherical waves by a wedge has been investigated. The integral representation of the exact solution chosen has two basic advantages. One is that the representations for the three types of waves are similar and consequently the expected accuracy can be conveniently compared. The other advantage is that the pattern functions appearing in the integrands of the diffraction integrals has its major contribution, irrespective of the location of the field point, from the neighbourhood of the stationary-phase point. The accuracy for the cylindrical- and spherical-wave excitations is improved via convenient exponential approximations of the source functions in the diffraction integral.

More accurate rational approximations to the Fresnel integral than that derived in this work do exist (Luke 1969). However, the special form derived in this work has the advantage of being simple and particularly useful for far-field analysis without requiring special care on crossing the shadow and reflection boundaries.

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